# Supplementary Material for "Learning the Structure of Deep Sparse Graphical Models"

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## 1 Proof of General CIBP Termination

In the main paper, we discussed that the cascading Indian buffet process (CIBP) for fixed and finite $\alpha$ and $\beta$ eventually reaches a restaurant in which the customers choose no dishes. Every deeper restaurant also has no dishes. Here, we show a more general result, for IBP parameters that vary with depth: $\alpha^{(m)}$ and $\beta^{(m)}$.

Let there be an inhomogeneous Markov chain $\mathcal{M}$ with state space $\mathbb{N}$. Let $m$ index time and let the state at time $m$ be denoted $K^{(m)}$. The initial state $K^{(0)}$ is finite. The probability mass function describing the transition distribution for $\mathcal{M}$ at time $m$ is given by the following equation:

$$p(K^{(m+1)} = k \,|\, K^{(m)}, \alpha^{(m)}, \beta^{(m)}) = \frac{1}{k!} \exp\left\{ -\alpha^{(m)} \sum_{k'=1}^{K^{(m)}} \frac{\beta^{(m)}}{k' + \beta^{(m)} - 1} \right\} \left( \alpha^{(m)} \sum_{k'=1}^{K^{(m)}} \frac{\beta^{(m)}}{k' + \beta^{(m)} - 1} \right). \quad (1)$$

**Theorem 1.1.** *If there exists some $\bar{\alpha} < \infty$ and $\bar{\beta} < \infty$ such that $\forall m$, $\alpha^{(m)} < \bar{\alpha}$ and $\beta^{(m)} < \bar{\beta}$, then $\lim_{m\to\infty} p(K^{(m)} = 0) = 1$.*

*Proof.* Let $\mathbb{N}^+$ be the positive integers. The $\mathbb{N}^+$ are a communicating class for the Markov chain (it is possible to reach any member of the class from any other member) and each $K^{(m)} \in \mathbb{N}^+$ has a nonzero probability of transitioning to the absorbing state $K^{(m+1)} = 0$, i.e., $p(K^{(m+1)} = 0 \,|\, K^{(m)}) > 0$, $\forall K^{(m)}$. If, conditioned on nonabsorption, the Markov chain has a stationary distribution (is *quasi-stationary*), then it reaches absorption in finite time with probability one. This is the requirement that, conditioned on having not yet reached a restaurant with no dishes, the number of dishes in deeper restaurants will not explode.

The quasi-stationary condition can be met by showing that $\mathbb{N}^+$ are positive recurrent states. We use the Foster–Lyapunov stability criterion (FLSC) to show positive-recurrency of $\mathbb{N}^+$. The FLSC is met if there exists some function $\mathcal{L}(\cdot) : \mathbb{N}^+ \to \mathbb{R}^+$ such that for some $\epsilon > 0$ and some finite $B \in \mathbb{N}^+$,

$$\sum_{k=1}^{\infty} p(K^{(m+1)} = k \,|\, K^{(m)}) \left( \mathcal{L}(k) - \mathcal{L}(K^{(m)}) \right) < -\epsilon \text{ for } K^{(m)} > B \quad (2)$$

$$\sum_{k=1}^{\infty} p(K^{(m+1)} = k \,|\, K^{(m)})\, \mathcal{L}(k) < \infty \text{ for } K^{(m)} \leq B. \quad (3)$$

For Lyapunov function $\mathcal{L}(k) = k$, the first condition is equivalent to

$$\left( \alpha^{(m)} \sum_{k=1}^{K^{(m)}} \frac{\beta^{(m)}}{k + \beta^{(m)} - 1} \right) - K^{(m)} < -\epsilon. \quad (4)$$

We observe that

$$\alpha^{(m)} \sum_{k=1}^{K^{(m)}} \frac{\beta^{(m)}}{k + \beta^{(m)} - 1} < \bar{\alpha} \sum_{k=1}^{K^{(m)}} \frac{\bar{\beta}}{k + \bar{\beta} - 1}, \quad (5)$$

for all $K^{(m)} > 0$. Thus, the first condition is satisfied for any $B$ that satisfies the condition for $\bar{\alpha}$ and $\bar{\beta}$. That such a $B$ exists for any finite $\bar{\alpha}$ and $\bar{\beta}$ can be seen by the equivalent condition

$$\left(\bar{\alpha}\sum_{k=1}^{K^{(m)}}\frac{\bar{\beta}}{k+\bar{\beta}-1}\right) - K^{(m)} < -\epsilon \text{ for } K^{(m)} > B. \tag{6}$$

As the first term is roughly logarithmic in $K^{(m)}$, there exists some finite $B$ that satisfies Eqn 6. The second FLSC condition is trivially satisfied by the observation that Poisson distributions have a finite mean. □